Bayesian Asymptotics

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Summary

The main goal of statistical estimation is to recover an unknown, fixed parameter of interest from noisy observations, which is achieved by an estimation procedure. In this thesis we consider the Bayesian approach to statistical inference by assigning a prior distribution to the unknown parameter. Next the corresponding posterior distribution serves as a starting point for estimation. If the parameter of interest is infinite-dimensional (nonparametric statistics), the choice of the prior is of significant importance and might dramatically influence the performance of the corresponding posterior.

In Chapter 1 we introduce important notions of Bayesian asymptotics: posterior consistency and posterior contraction, (frequentist) coverage of credible balls, and posterior limits. Even though the existing literature on Bayesian nonparametrics is large, several important aspects of the models considered in this thesis have not been studied so far. We briefly introduce nonparametric inverse problems and show why the general theory of posterior contraction cannot be applied in this setting. We also present the classical Bernstein–von Mises theorem and review the recent developments in the study of posterior limits in semi- and nonparametric statistical problems.

In Chapter 2 we first describe a nonparametric inverse problem in the context of the canonical signal-in-white noise model with the operator acting between two Hilbert spaces, and show its equivalence to the infinite-dimensional normal mean model. The main contribution of this chapter is the study of the asymptotic properties of the posterior in two settings of inverse problems: mildly and extremely ill-posed. The former setting covers, among others, estimation of a derivative of a function, and the latter is presented by a study of the recovery of the initial condition for the heat equation. We consider a certain family of Gaussian prior distributions and show that the rate of contraction depends on the parameters of the prior, characteristics of the inverse problem, and the regularity of the true parameter of interest. These results are compared with the existing frequentist approaches to nonparametric inverse problems. We also discuss frequentist properties of Bayesian credible balls. The results on contraction and credibility are illustrated by simulation examples in both inverse problem settings.

In Chapter 3 we present the first theoretical study of adaptive Bayesian procedures for nonparametric inverse problems. Again, as in Chapter 2, we consider a certain family of Gaussian priors for the parameter of interest. These priors are indexed by a parameter α quantifying the "regularity" of the prior. In Chapter 2 we considered this parameter fixed, and in this chapter we select α using the data. A first approach is fully Bayesian: we endow the parameter α with a prior distribution itself. A second approach we study mixes the Bayesian and the frequentist paradigm: we first "estimate" α from the data in a frequentist manner, and then substitute the estimator $\hat{\alpha}_n$ for α in the posterior distribution obtained in Chapter 2. We show that both methods lead to adaptation and rate-optimality (up to lower order factors) over two families of submodels containing the true parameter of interest, and describing its regularity. We illustrate both methods by the simulation example introduced in Chapter 2 in the mildly ill-posed inverse problem setting.

In Chapter 4 we consider a semiparametric aspect of inverse problems: recovery of linear functionals of the parameter of interest. We consider not only continuous, but also certain discontinuous functionals, belonging to a wider class of prior-measurable linear functionals. The contribution of this chapter is similar to the one of Chapter 2: we study posterior contraction that is not covered by the existing literature on the subject, and we investigate the frequentist coverage of Bayesian credible intervals. The regularity of the linear functional plays an important role in the asymptotic behavior of Bayesian procedures. We show that certain continuous linear functionals cancel the inverse nature of the problem, and put the problem in the regular regime. In this chapter we obtain a semiparametric Bernstein–von Mises theorem, not only with a typical $n^{-1/2}$ rate, but also with a rate slowed down by a slowly varying factor. The results of this chapter are illustrated by the same simulation examples as in Chapter 2.

In Chapter 5 we first present a simple irregular model, consisting of shifted exponential distributions with scale 1, and consider the resulting posterior limit. Next we introduce *local asymptotic exponentiality (LAE)*, an irregular expansion of the likelihood, presented in the semiparametric setting in which we decompose the parameter as a pair (θ, η), where the parameter of interest θ lies in an open subset of the real line, and the nuisance parameter η is an element of an infinite-dimensional space. We next present the main theorem of the chapter, an irregular version of a semiparametric Bernstein–von Mises theorem. A separate section of the chapter is dedicated to the most demanding condition of the main theorem, namely marginal consistency at n^{-1} rate. Some discussion is provided, followed by a lemma verifying the condition based on a condition on the likelihood ratio. We end the chapter by presenting two semiparametric models exhibiting the LAE property. Both problems are related to the problem of estimation of the boundary point of a distribution. The first is a generalization of the shifted exponential model. The other one generalizes the uniform distribution on the interval $[0, \theta]$. In both settings, based on an i.i.d. sample distributed according to an unknown, but fixed element (θ_0, η_0), we obtain exponential limits for the marginal posterior distributions.